AGA

Algebra 1 Geometry Algebra 2

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PROFESSIONAL DEVELOPMENT

Mathematical Processes and Practices

Developing Proficiencies in Mathematics Learners

According to Principles to Actions (National Council of Teachers of Mathematics, 2014), "An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically" (p. 5). What this means for high school students and how to engage students in this sort of meaningful learning are addressed in the following article.

There are eight mathematical processes and practices. They are based on the National Council of Teacher of Mathematics' (NTCM) Process Standards (NCTM, 2000) and the National Research Council's (NRC) Strands of Mathematical Proficiency (NCR, 2001). Students who are engaged in the mathematical processes and practices around important mathematics are likely engaged in meaningful learning as described in Principles to Actions.

It is likely that good teachers can find evidence of each of the mathematical processes and practices in their current teaching. Regardless, it is useful to examine them and think about how each contributes to the development of mathematically proficient students. What follows is a description of how they might look in a high school classroom. The examples discussed here are reflective of experiences supported by HMH *Algebra 1*, HMH *Geometry*, and HMH *Algebra 2*.

HMH *Algebra 1*, HMH *Geometry*, and HMH *Algebra 2* support the mathematical processes and practices through specific features, such as the following:

- Explorations at the start of lessons engage students in seeing structure and generalizing.
- Reflect and Elaborate questions prompt students to express their understanding of concepts using precise mathematical language.
- Performance Tasks at the lesson, module, and unit levels offer students extended problem-solving and abstract-reasoning opportunities.
- Real-world and H.O.T. problems involve students in mathematical modeling as well as logical and quantitative reasoning.

Mathematical Processes and Practices

Developing Proficiencies in Mathematics Learners

MPP1: Problem Solving

In order for students to develop the diligence required for problem solving, they must be provided with problems for which a pathway toward a solution is not immediately evident. For instance, as part of the study of probability in high school math, students can be presented with the "Problem of Points," a classic problem discussed and solved by the mathematicians Blaise Pascal and Pierre de Fermat in an exchange of letters in the 1600s.

The problem involves the fair division of the stakes put up by two people who play a game of chance that is interrupted. Suppose the players each put up $10 and agree to the following rules:

1. They take turns flipping a coin.
2. Player A gets 1 point each time the coin shows heads.
3. Player B gets 1 point each time the coin shows tails.
4. The player who reaches 10 points first wins all the money.

Now suppose the game is interrupted when player A has 8 points and player B has 6 points. How should the money be divided between the players?

Although there are many ways to answer the question (all of which should be judged in terms of their fairness to the players), Pascal and Fermat approached the problem by enumerating the possible ways that the game could end and divided the money between the players in a ratio based on the number of ways each player could win. Students can follow in Pascal and Fermat's footsteps toward a solution of the problem with just the basic understanding that each outcome of a flip of a (fair) coin has a probability of $\frac{1}{2}$.

MPP2: Abstract and Quantitative Reasoning

When solving a real-world problem, students must reason abstractly about the variables involved in the problem, and they must reason quantitatively with the units of measurement given in the problem. For instance, consider the following problem:

Maria can run 100 yards in 30 seconds, while Sophia can run 1 mile in 10 minutes. The two agree to run a 5 mile race. Because Maria thinks that she is the faster runner, she gives Sophia a 5 second head start. Who wins the race? Explain your reasoning.

Students need to recognize that distance and time are the variables in this problem. Using t to represent time, students must decide what units of time (seconds, minutes, or hours) to use for t and whether time will be measured from the moment Maria begins to run or from the moment that Sophia begins to run. Those choices in turn affect each expression that students write to represent the time that Maria and Sophia are running and how those expressions relate to one another.

Quantitative reasoning is also needed when students move on to calculating distances. First, using d to represent distance, they must decide what units of distance (feet, yards, or miles) to use for d. Then, they must convert the rates stated in the problem to equivalent rates that are compatible with the units of distance and time they selected in order to apply the formula $d = rt$. Only at that point can students either set the two expressions for distance equal or graph the two distance functions to see where the graphs intersect. Once students have determined a solution, they need to interpret that solution within the given context in order to determine who wins the race.

MPP3: Use and Evaluate Logical Reasoning

Students need to explain and justify their solution strategies. They should also listen to the explanations of other students and try to make sense of them. They will then be able to incorporate the valid reasoning of others into their own strategies and improve upon their own solutions. For instance, consider the following description of a student exploration.

Students are asked to approximate the volume of a right circular cone having a base radius of 1 unit and a height of 2 units. Students are told not to use the formula for the volume of a cone but instead use circular cylinders and the formula for the volume of a circular cylinder. One student proceeds by drawing stacked cylinders each having a height of 0.5 unit. Using an argument based on the similar right triangles in a cross section of the cone and overlaid stacked cylinders, the student concludes that the radius of the cone at any given distance from the vertex of the cone is ½ of that distance, as shown in Table 1. The student then averages consecutive radii in Table 1 and uses those averages as the radii of the stacked cylinders in order to find the volumes of the cylinders, as shown in Table 2.

Table 1	
Distance from vertex of cone	**Radius of cone at that distance**
0	0
0.5	0.25
1	0.5
1.5	0.75
2	1

Table 2		
Cylinder radius	**Cylinder height**	**Cylinder volume**
$\frac{0 + 0.25}{2} = 0.125$	0.5	$\pi(0.1252)(0.5) \approx 0.025$
$\frac{0.25 + 0.5}{2} = 0.375$	0.5	$\pi(0.3752)(0.5) \approx 0.221$
$\frac{0.5 + 0.75}{2} = 0.625$	0.5	$\pi(0.6252)(0.5) \approx 0.614$
$\frac{0.75 + 1}{2} = 0.875$	0.5	$\pi(0.8752)(0.5) \approx 1.203$

The student concludes that an approximation of the volume of the cone using the volumes of the stacked cylinders is 0.025 + 0.221 + 0.614 + 1.203, or about 2.063 cubic units.

This student has constructed a viable argument using four stacked cylinders. Another student agrees that this approach works and suggests that an even better approximation of the volume of the cone could be obtained by using more than four stacked cylinders. This second student has made sense of the reasoning of the first student and offers a conjecture of a way to improve on the results.

MPP4: Mathematical Modeling

Mathematical modeling involves expressing real-world relationships and data in mathematical terms in order to gain insight into a real-world situation. Students must employ analytical skills throughout the multi-step process of creating a mathematical model, investigating the model, and drawing conclusions from the model. The types of questions that students must ask themselves include the following:

1. What are the variables or unknown information in this real-world situation? What information is given in this situation?
2. What relationship does the unknown information appear to have with the given information? What suggests or confirms this relationship?
3. How do I express the model mathematically? That is, what expressions, equations, or inequalities can I write, or what drawings or graphs can I make?
4. What does the model tell me about the real-world situation that I don't already know?
5. Do the conclusions that I draw from the model make sense in the context of the real-world situation? If not, is the model seriously flawed or merely limited due to simplifying assumptions?

For instance, given a table of real-world data, such as the population of a state over time, students would recognize the independent variable (time) and the dependent variable (population). They might draw a scatter plot in order to look for a pattern in the data.

The pattern might suggest a linear function, an exponential function, or some other type of function. Students would then fit a line or curve to the data and check it for goodness of fit. After obtaining an equation for the fitted function, students could use it to make predictions, such as the population's average rate of change or the time until the population reaches a certain value, and then check the reasonableness of the predictions against reality. Unreasonable results may lead to refinement or reconsideration of the model, or simply an acknowledgement that there may be lurking (or unknown) variables in the data, making predictions from the model unreliable.

MPP5: Use Mathematical Tools

At the high school level, mathematical tools run the gamut from physical tools such as manipulatives, concrete models, rulers and protractors, and compasses and straightedges to technological tools such as graphing technology, spreadsheets, and computer algebra systems. Of course, the most commonly used of all mathematical tools are simple paper and pencil. Students should be flexible in their use of these tools, taking into account the capabilities and limitations of the tools when selecting them.

For instance, students who are familiar with the Rational Root Theorem should recognize that the polynomial equation $2x^4 + 5x^3 + 5x^2 + 20x - 12 = 0$ has 16 possible rational roots: , ±½,1, ±³⁄₂, ±2, ±3, ±4, ±6, ±12. While students can certainly check these rational roots using the paper-and-pencil method of synthetic division, they can instead use a graphing calculator to narrow the list of possibilities by graphing the function $f(x) = 2x^4 + 5x^3 + 5x^2 + 20x - 12$ and observing the graph's x-intercepts. The x-intercepts suggest that two of the equation's roots are −3 and ½. Students must still confirm that these are roots by using synthetic division, which leaves the quotient $2x^2 +8$. At this point, a graphing calculator is no longer useful, because the graph can reveal only real roots. Students must now solve the equation $2x^2+8=0$ over the set of complex numbers to get the remaining two roots of the original equation: ±2i.

MPP6: Use Precise Mathematical Language

Using precise language, especially when stating definitions, postulates, theorems, and properties, is fundamental to doing mathematics. Even a slight change in a mathematical statement can have significant consequences. For instance, consider these two alternative definitions of trapezoid:

- Exclusive definition: A trapezoid is a quadrilateral having exactly one pair of parallel sides.
- Inclusive definition: A trapezoid is a quadrilateral having at least one pair of parallel sides.

The first definition is called exclusive because it excludes parallelograms (quadrilaterals having two pairs of parallel sides) as a special type of trapezoid. Obviously, the second definition includes parallelograms as a special type of trapezoid. These definitions result in different configurations of the quadrilateral hierarchy: The exclusive definition treats trapezoids and parallelograms as distinct sets, whereas the inclusive definition treats the set of parallelograms as a subset of the set of trapezoids.

When parallelograms are treated as a special type of trapezoid, any property of trapezoids automatically applies to parallelograms.
For instance, by breaking a trapezoid into a rectangle and two right triangles, you can derive the area formula $A=½(b_1+b_2),h_1$ where b_1

and b^2 are the lengths of a pair of parallel sides (called bases) and h is the distance (or height) between the parallel sides. With an inclusive definition of trapezoid, this same area formula applies to parallelograms. In other words, you do not need to derive a separate area formula for parallelograms.

MPP7: See Structure

When students see structure in mathematical objects (expressions, equations, figures, and so on), they are able to exploit that structure to accomplish a mathematical goal. Here are some examples:

- The student who is asked to find the product $(x + 1)(x + 2)$ using the distributive property $a(b + c)=ab+ac$ must think of the binomial $x + 1$ as a single entity that can be distributed to the terms of the binomial $x + 2$, resulting in the expression $(x + 1)x +(x + 1)2$, which can in turn be simplified by applying the distributive property two more times to produce the result $x^2 + x + 2x + 2$, or $x^2 + 3x + 2$.
- The student who is asked to graph the quadratic function $f(x)= 2x^2 - 4x+5$ can complete the square to write the function as $f(x)= 2(x-1)^2+3$. In this form, the student can see the symmetry of the graph about the vertical line $x = 1$ because $f(1+p) = f(1-p) = 2p^2 +3$ for every value of p. This allows the student to plot a point on the graph on one side of the line $x = 1$ and know that the point's reflection across the line is also on the graph.
- The student who knows how to construct a regular hexagon by using a compass to draw a circle, then using the same compass setting to mark six congruent arcs along the circle, and then drawing line segments to connect the endpoints of those arcs should see that connecting every other endpoint instead results in the construction of an equilateral triangle. Moreover, drawing two rays from the center of the circle to the endpoints of one of the six congruent arcs, then bisecting the angle formed by the rays, and then using one of the halves of the bisected arc to mark the midpoint of each of the other arcs, the student should see that connecting all 12 points (6 endpoints and 6 midpoints) results in the construction of a regular dodecagon.

MPP8: Generalize

Whether practicing routine skills or solving complex problems, students should be encouraged to look for, and take advantage of, the regularity of mathematics. For instance, after students solve a series of two-step equations of the form $ax + b = c$ (where $a \neq 0$, of course), you might introduce them to the literal equation $ax + b = c$ and have them solve it for x, resulting in $x = c-b/a$. This general solution can be applied to any specific equation, such as $3x + 2 = 17$, which has the solution $x = 17 - 2/3 = 5$. In fact, this approach is the idea behind the quadratic formula, $x = -b\pm \sqrt{b^2-4ac} /2a$, which gives the solutions of any quadratic equation of the form $ax^2 +bx+c = 0$.

Another example of the role that regularity plays in mathematics is when rational exponents are defined. With properties of exponents already established for integer exponents, it's reasonable to extend those properties to rational exponents. So, in order to define what the number $3^{1/2}$ represents, mathematicians applied the Power of a Power Property to the square of $3^{1/2}$ and got $(3^{1/2})^2 = 3^{1/2 \cdot 2} = 3^1 = 3$. By definition, if a nonnegative number a is the square of a number b then b is a square root of a. So, treating $3^{1/2}$ as another way to write $\sqrt{3}$ guarantees consistency with the properties of exponents.

Supporting Mathematical Processes and Practices Through Questioning

While some students may develop proficiency with the mathematical processes and practices on their own, most will need guidance and support from the teacher. One important way to provide that guidance and support is to integrate questioning strategies that help students focus on the processes and practices. The table below gives examples of questioning strategies.

When you ask...	**Students...**
• What is the problem asking? • How will you use that information? • What other information do you need? • Why did you choose that strategy? • What is another way to solve the problem? • What did you do first? Why? • What can you do if you don't know how to solve a problem? • Have you solved a problem similar to this one? • When did you realize that your first method would not work for this problem? • How do you know whether your answer makes sense?	Use problem solving.
• What is a situation that could be represented by this equation/inequality? • What expression represents this situation? • Why does that expression represent the situation? • What properties or theorems did you use to find the answer? • How do you know whether your answer is reasonable?	Use abstract and quantitative reasoning.
• Will that method always work? • How do you know? • What do you think about what she said? • Who can tell us about a different method? • What do you think will happen if...? • When would that not be true? • Why do you agree/disagree with what he said? • What do you want to ask her about that method? • How does that drawing support your work?	Use and evaluate logical reasoning.

When you ask...	Students...
• Why is that a good model for this problem? • How can you use a simpler problem to help you find the answer? • What conclusions can you make from your model? • How would you change your model if...?	Use mathematical modeling.
• What could you use to help you solve the problem? • What strategy could you use to make that calculation easier? • How would estimation help you solve the problem? • Why did you decide to use...?	Use mathematical tools.
• How can you use math vocabulary in your explanation? • How do you know whether those answers are equivalent? • What does that mean?	Use precise mathematical language.
• How did you discover that pattern? • What other patterns can you find? • How does that property apply to this problem? • How is that like...?	See structure.
• What do you remember about...? • What happens when...? • What if you...instead of...? • What might be a shortcut for...?	Generalize.

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